

On the Implications of Discrete Symmetries for the β -function of Quantum Hall Systems

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We argue that the large discrete-symmetry group of quantum Hall systems is insufficient in itself to determine the complete β -function for the scaling of the conductivities, σ_{xx} and σ_{xy} . We illustrate this point by showing that a recent ansatz for this function is one of a many-parameter family. A clean prediction for the delocalization exponents for these systems therefore requires the specification of more information, such as past proposals that the β -function is either holomorphic or quasi-holomorphic in the variable $z = (\hbar/e^2)(\sigma_{xy} + i\sigma_{xx})$.

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I. INTRODUCTION

It has been conjectured [1] that the large group of discrete symmetries enjoyed by quantum Hall systems [2,3] might permit the complete determination of the β -function of these materials. This β -function describes the renormalization-group (RG) flow of the conductivities σ_{xx} and σ_{xy} in the delocalization-scaling theory of these systems. The question addressed in ref. [1] was whether a few physically well-motivated constraints on the β -function, exploiting available data both from the weakly and the strongly coupled domains of the system, suffice to completely fix the full non-perturbative form of the β -function in an appropriate scheme, without actually deriving it from a microscopic theory.

More precisely, the main idea was that the consistency of the RG flow with the discrete symmetry group — which relates large and small σ_{xx} — might be sufficient to completely determine the β -function if combined with the known weak-coupling results in the asymptotic domain of large σ_{xx} . Such a construction would specify, among other quantities, the universality class (and critical exponents) of the quantum critical “anyon delocalization” points which are believed to exist in this system. Since this goal was largely thwarted by the existence of more than one β -function which satisfied these conditions, the main focus of ref. [1] is the establishment of further conditions which would uniquely determine a solution.

There has been a resurgence of interest in this topic of late, with the recent appearance of two ansätze [4,5] for the nonperturbative form of this exact β -function. In this letter we add our own multiple-parameter ansatz to this list. We revive this ansatz not because we believe it to be the final word on this subject, but rather to emphasize that the determination of the β -function requires more than simply symmetry and asymptotic information, as well as to clarify the relation of these ansätze with the conditions outlined in ref. [1].

In ref. [1] it was the necessity for extra information

which drove the formulation of quasi-holomorphy, while in ref. [4] it is holomorphy which is invoked for the same reasons. It would be interesting to understand what this information might be for the ansatz of ref. [5], which is sufficiently narrow to produce specific values for the delocalization critical exponents. It is otherwise difficult to know why this ansatz should be preferable over others which may also do so.

We present our ideas in the following way. First, we briefly recap the symmetry and asymptotic conditions which we demand of the β -function. In so doing we expand on the consequences of holomorphy, in order to make contact with ref. [4]. Next, we describe our ansatz, which contains that of ref. [5] as a special case. We use this ansatz to explore in more detail the form taken by β at large σ_{xx} and near the critical points.

II. SYMMETRIES AND ASYMPTOTICS

The precise form of the constraints identified in ref. [1] are:

1. *Asymptotics:* The β -function which is predicted by the effective sigma-model of weak localization in a magnetic field (see ref. [7] for a comprehensive review) may be explicitly computed for large σ_{xx} , since this corresponds to weak coupling in the sigma-model. Writing $\sigma_{xy} + i\sigma_{xx} = (e^2/\hbar) z$ where $z = x + iy$ and $\bar{z} = x - iy$, we have:

$$\beta^z = \frac{dz}{dt} = b_0 + \frac{b_1}{y} + \frac{b_2}{y^2} + \dots + (q, \bar{q} - \text{expansion}). \quad (1)$$

Here t is a logarithmic scale parameter, $q = \exp(2\pi iz)$, $\bar{q} = \exp(-2\pi i\bar{z})$, and the “ q, \bar{q} -expansion” refers to the leading non-perturbative (dilute instanton gas) corrections to the perturbative loop expansion in $1/y$.

The coefficients b_i of the perturbative sigma-model β -function are known through six-loop order ($i \leq 4$) [6], with values:

$$b_0 = 0, \quad b_1 = -\frac{i}{2\pi^2}, \quad b_2 = 0, \quad b_3 = -\frac{3i}{8\pi^4}, \quad b_4 = 0. \quad (2)$$

The even powers of $1/y$ vanish in this renormalization scheme (to the order known). Note, however, that only the leading non-vanishing coefficient b_1 is scheme independent. By contrast, only the leading term in the instanton expansion is available [8], and is proportional to the anti-instanton result: $y^k \bar{q}$, for k positive.

There are three senses in which the β -function might be asked to agree with eqs. (1) and (2).

1. *Weak Agreement:* The weakest condition simply requires agreement with the scheme-independent perturbative part of these results — *i.e.* with only b_0 and b_1 .
2. *Strong Agreement:* A stronger requirement is agreement with the entire perturbative expansion, as computed in the sigma-model.
3. *Very Strong Agreement:* The strongest requirement would be agreement with all known terms, *including* the dilute instanton gas expansion. None of the ansätze which have been proposed to date satisfy this condition, since they do not properly reproduce the power of y in the leading instanton result. We shall not here focus much attention on this condition, however, since existing instanton calculations for disordered systems are performed using the replica trick, which is known to sometimes fail even for systems where the perturbative results they give are accurate [9].

2. *Scaling:* We require the β -function to share the (real) analytic structure to be expected of any RG flow. That is, since β^z describes the differential elimination of degrees of freedom at a particular scale, it does not contain singularities, apart from those places where the number of relevant degrees of freedom change. We therefore demand that the β -function should be non-singular throughout the upper half of the complex z -plane ($\mathbb{H} : \text{Im} z > 0$). The continuous (second order) quantum critical delocalization transitions in this system (see ref. [10] for a review) are then identified with the zeros of β . From eqs. (1) and (2) it is clear that β also goes to zero as $z \rightarrow i\infty$.

Because $\sigma_{xy} = (e^2/\hbar) x$ enters the sigma model as the coefficient of a topological term, β cannot depend on x to any order in perturbation theory, implying that the leading asymptotic behaviour of β is a function of y alone. Thus, unless all b_i except b_0 vanish (as happens in systems with unbroken complex supersymmetry), agreement with the sigma-model large- y form precludes the β -function being a complex analytic (holomorphic) function of z only.

The exploitation of scaling ideas and data are at the very heart of the “phenomenological” approach proposed in ref. [2], and further pursued in ref. [1]. It is universality which allows us to entertain the idea that the β -function could be determined up to asymptotics by macroscopic scaling properties. Conversely, should the β -function be determined, it need not shed much light on the nature of the microscopic physics responsible for the scaling laws in this system. It can, however, show that all the scaling data, as well as the phase diagram and Hall quantization, are encoded in the low-energy effective theory as a global discrete symmetry of the (complexified) Kramers-Wannier type. It is this alleged symmetry which provides the final physically-motivated constraint on the β -function. It also is what endows our conjecture with most of its power, relating properties which are perturbative within the sigma-model context to those which are not.

3. *Automorphy:* As explained at length elsewhere [2], the observed “superuniversality” of the critical exponents in the hierarchy of delocalization transitions that take place in the quantum Hall system was the original motivation for conjecturing that the low-energy theory respects, in the fully spin-polarized case, a global discrete symmetry $\Gamma = \Gamma_0(2)$. ($\Gamma_0(2)$, which is also denoted $\Gamma_T(2)$ in refs. [1,2], is a well-known sub-group of the modular group, $SL(2, \mathbb{Z})$.) The mathematical fact that this sub-group automatically ensures Hall quantization on odd-denominator fractions (when $\sigma_{xx} \rightarrow 0$) is also encouraging. Independent arguments arose at about the same time in the form of the “law of corresponding states”, from a mean-field treatment of the microscopic theory [3].

The modular group (and its subgroups) act on the complex conductivity as special Möbius transformations: $\gamma(z) = (az + b)/(cz + d)$ where a, b, c, d are integers satisfying $ad - bc = 1$. The subgroup $\Gamma_0(2)$ is defined by the additional condition that c be even.

If such a symmetry is present at low energy the β -function must respect it in a very specific sense. A function $f(z, \bar{z})$ is called automorphic of weight (u, v) under Γ *iff* it transforms like a generalized tensor:

$$f(\gamma(z), \gamma(\bar{z})) = \left(\frac{d\gamma}{dz}\right)^{-u/2} \left(\frac{d\bar{\gamma}}{d\bar{z}}\right)^{-v/2} f(z, \bar{z}) \\ = (cz + d)^u (c\bar{z} + d)^v f(z, \bar{z}) \quad (3)$$

for every γ in Γ .

It was shown in ref. [1] that if the RG commutes with Γ , then the physical (contravariant) β -function β^z is a *negative* weight $(-2, 0)$ function, while the complex conjugate function, $\beta^{\bar{z}}$, has weight $(0, -2)$. Similarly, given a metric G_{ij} , the covariant β -function, $\beta_i = G_{ij} \beta^j$ must have positive weights: $\beta_z \sim (2, 0)$ and $\beta_{\bar{z}} \sim (0, 2)$.

Because the constraints 1 through 3 are extracted from

experimental data and/or general knowledge about scaling and perturbation theory, they would seem to be a reasonable starting point for the search for the exact quantum-Hall β -function. Since we display many solutions to these conditions below they cannot be sufficient in themselves to uniquely determine the result.

Before presenting these solutions we pause to discuss the holomorphy and quasiholomorphy assumptions.

III. HOLOMORPHY

A natural guess for β^z is that it is a holomorphic (or anti-holomorphic) function: $\beta^z = \beta^z(z)$ (or $\beta^z = \beta^z(\bar{z})$), a proposal recently revived in ref. [4]. This is a very predictive ansatz because it permits the use of powerful results from complex analysis. As discussed above, this ansatz is inconsistent with even the weak form of the sigma-model behaviour at large y , and so it necessarily implies the breakdown of this sigma-model description of weak localization in magnetic fields. The purpose of the present section is to establish that a holomorphic (or anti-holomorphic) β -function *must* also have a singularity somewhere in $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \infty$, where \mathbb{Q} are the rational numbers. One could conceivably tolerate a pole on the real axis, but it is then difficult to obtain an acceptable flow. For these two reasons the holomorphic option was rejected in ref. [1].

A particularly useful fact for any meromorphic function f of weight $(k, 0)$ with respect to $\Gamma_0(2)$, relates the ‘index’ of its zeros and poles within a fundamental domain $\overline{\mathbb{F}}$ of $\overline{\mathbb{H}}$ [11]:

$$n_\infty + n_0 + \frac{n_*}{2} + \sum_p n_p = \frac{k}{4}. \quad (4)$$

Here n_p is the leading power of $z - z_p$ which appears in a Laurent expansion of f about the pole or zero at z_p in the interior or boundary of $\overline{\mathbb{F}}$. n_* is the same quantity in the expansion of f about the fixed point, $z_* = (1 + i)/2$, of the group $\Gamma_0(2)$. Similarly, n_∞ is the leading power in a Laurent expansion of f in powers of q about $z = i\infty$, while n_0 counts the leading power of $\tilde{q} = \exp(-i\pi/z)$ in an expansion of $z^k f(z)$ in powers of \tilde{q} about $z = 0$ [12,13].

Eq. (4) implies, in particular, that no weight $(-2, 0)$ function like β^z can be holomorphic without acquiring singularities somewhere in $\overline{\mathbb{H}}$. For example, as was observed in ref. [14], the β -function for the Seiberg-Witten $N = 2$ supersymmetric $SU(2)$ gauge theory is the unique weight $(-2, 0)$ function (up to normalization) which has a simple zero only at z_* (and its images under $\Gamma_0(2)$) and which approaches a constant as $z \rightarrow i\infty$. The complex quantity z in this case is related to the gauge coupling (g) and vacuum angle (θ) by: $z = (\theta/2\pi) + (4\pi i/g^2)$. Eq. (4) forces $\beta(z)$ to have a simple pole at $z = 0$ (as well as

the other integers on the real axis), corresponding to the infinite-coupling limit in the sigma-model.

Similar considerations apply if the quantum Hall β -function were to be holomorphic, and in particular it would also be singular somewhere. A simple proposal is to choose β^z or $\beta^{\bar{z}}$ to be proportional to the Seiberg-Witten $N = 2$ supersymmetric β -function. Unfortunately, the flow in this case is *repelled* by the odd-denominator fractions on the real axis, instead flowing towards z_* , which is an attractive fixed point, with no irrelevant directions. Clearly this flow cannot describe the second-order transitions of the quantum Hall systems.

One might imagine making more complicated choices, such as to force the β -function to have simple zeroes (*i.e.* $n_\infty = n_* = 1$) both at $i\infty$ and z_* , with no poles or singularities elsewhere for nonzero σ_{xx} — thus making it at least qualitatively similar to the perturbative sigma-model result. Such a condition must have a double pole at $z = 0$. We now argue that *any* holomorphic β -function having a simple zero at $z = z_*$ cannot have both a relevant and irrelevant direction there, giving an unacceptable flow.

To establish this result proceed as follows. The critical exponents are related to the derivative of the β -function at its zeroes, and a simple argument shows that holomorphy dictates that these must have the same sign. This is because the the matrix of derivatives for holomorphic β^z necessarily has the following form:

$$\begin{pmatrix} \frac{d\beta^z}{dz} & \frac{d\beta^z}{d\bar{z}} \\ \frac{d\beta^{\bar{z}}}{dz} & \frac{d\beta^{\bar{z}}}{d\bar{z}} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & \bar{B} \end{pmatrix}, \quad (5)$$

from which we see that the product of the eigenvalues of this matrix is $B\bar{B} \geq 0$, implying: (i) both eigenvalues have the same sign; or (ii) one (or both) is zero. Neither of these cases describes the observed flow near the quantum Hall critical points.

As observed in refs. [4,15], qualitatively acceptable flows are obtained by moving the pole of the β -function to the fixed point z_* , in which case the β -function can approach a constant as $z \rightarrow i\infty$ and $z \rightarrow 0$. The simplest such β -function — which has $n_* = -1$ and all others zero — turns out to be just the inverse of the holomorphic weight $(2, 0)$ function $\mathcal{E}(z)$ defined below. As discussed in ref. [4], the pole makes it problematic to identify the universal critical exponents at z_* , and ref. [4] instead explicitly exhibits the flow near this point in order to make comparisons with the data.

IV. QUASI-HOLOMORPHY

The alternative followed in ref. [1] was to start with the observation that eq. (4) is much more kind in its implications for the covariant function, β_z , than it is for β^z . This is because β_z transforms under $\Gamma_0(2)$ as an

automorphic function of *positive* weight ($k = 2$). In fact, for $\Gamma_0(2)$ — but not for $SL(2, \mathbb{Z})$ — there is a unique (up to normalization) holomorphic $(2, 0)$ function $\mathcal{E}(z)$, which is nowhere singular on \mathbb{H} . It can be expressed in terms of the famous modular discriminant function $\Delta = q \prod (1 - q^n)^{24}$ which generates the holomorphic (but *not* automorphic) Eisenstein function:

$$E_2(z) = \frac{1}{2\pi i} \partial_z \log \Delta = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad (6)$$

as follows:

$$\mathcal{E}(z) = 2 E_2(2z) - E_2(z) = 1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n}. \quad (7)$$

$\mathcal{E}(z)$ transforms as a weight $(2, 0)$ function with respect to $\Gamma_0(2)$ even though $E_2(z)$ does not. Unfortunately, since $\mathcal{E}(z)$ does not vanish as $z \rightarrow i\infty$ it is inconsistent with the perturbative expression, eq. (1).

The inconsistency between $\mathcal{E}(z)$ and the perturbative result in powers of $1/y$ motivates the search for more general weight $(2, 0)$ quantities which are not holomorphic but have the more general form: $1/y + g(z)$, for holomorphic $g(z)$. Such functions were called quasi-holomorphic in ref. [1].

The most general quasi-holomorphic $(2, 0)$ form which is nowhere singular in \mathbb{H} is a linear combination of $\mathcal{E}(z)$ and:

$$\begin{aligned} \mathcal{H}(z, \bar{z}) &= \frac{1}{\pi y} + \frac{2}{3} [E_2(2z) - E_2(z)] \\ &= \frac{1}{\pi y} + 16 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^{2n}}. \end{aligned} \quad (8)$$

Both \mathcal{E} and \mathcal{H} vanish at $z_* = (1 + i)/2$. It is the unlikely existence of this quasi-holomorphic “Hecke” function [16] which makes the idea of quasi-holomorphy useful. Further motivations for restricting attention to quasi-holomorphic building blocks are discussed in ref. [1].

4. Quasi-holomorphy: Ref. [1] therefore proposed that β_z is quasi-holomorphic. The unique quasi-holomorphic choice which is consistent with the weak form of agreement with sigma-model perturbation theory then is (if $G_{ij} \rightarrow 1$ as $z \rightarrow i\infty$):

$$\beta_z = \frac{i}{2\pi} \mathcal{H}. \quad (9)$$

With this proposal, the scheme dependence of β^z enters through the definition of the metric, G_{ij} .

V. MORE ANSÄTZE

If the β -function is not holomorphic then the previously-mentioned conditions are insufficient to com-

pletely pin it down. To establish this point, we now exhibit many more ansätze which satisfy all but the very-strong asymptotic condition.

Since the ratio of any two $(0, -2)$ functions is a $(0, 0)$ function, any $(0, -2)$ function $\beta^{\bar{z}}$ can be written as:

$$\beta^{\bar{z}} = \frac{i}{2\pi} W R, \quad (10)$$

where, $R(z, \bar{z})$ is a weightless (weight $(0, 0)$) function to be specified and, in the spirit of quasi-holomorphy, we choose to write the weight $(0, -2)$ factor as $W = \mathcal{H}/\mathcal{D}$ with:

$$\mathcal{D} = |\mathcal{E}|^2 + a\pi^2 |\mathcal{H}|^2 + \pi b \mathcal{E} \bar{\mathcal{H}} + \pi c \mathcal{H} \bar{\mathcal{E}} + \frac{d}{y^2}. \quad (11)$$

Here a, b, c and d are constants, and \mathcal{D} transforms under $\Gamma_0(2)$ as a weight $(2, 2)$ function.

In order to make the invariance of R with respect to $\Gamma_0(2)$ explicit, it is convenient to change variables from z to $f = -\vartheta_3^4(z) \vartheta_4^4(z) / \vartheta_2^8(z)$, and write $R = R(f, \bar{f})$. This may always be done since this f plays the same role for $\Gamma_0(2)$ as Klein’s famous j -function does for the full modular group. In particular, it is invariant under $\Gamma_0(2)$ and uniquely labels every point in the fundamental domain \mathbb{F} of the group (*i.e.* it is a one-to-one map of \mathbb{F} onto the complex sphere).

So far we have made no assumptions beyond automorphy. In choosing our ansatz for a, b, c, d and R our guidance is (strong) agreement with the large- y limit, as well as requiring a zero of $\beta^{\bar{z}}$ at z_* , and the absence of singularities and zeros elsewhere in \mathbb{H} .

Ansatz 1: The simplest case is to choose $d \neq 0$, in which case $\mathcal{D} > 0$ throughout \mathbb{H} so long as b and c are sufficiently small. In this case all assumptions are satisfied with the choice $R = 1$. Thus:

$$\begin{aligned} \beta^{\bar{z}}(z, \bar{z}) &= \frac{i}{2\pi} \frac{\mathcal{H}(z, \bar{z})}{\mathcal{D}(z, \bar{z})} \\ &= \frac{i}{2\pi^2 y} \left(1 + \frac{b+c}{y} + \frac{a+d}{y^2} \right)^{-1} + O(q, \bar{q}). \end{aligned} \quad (12)$$

Notice that $b_2 = 0$ in agreement with eq. (2) if we set $b = -c$ in the ansatz, in which case $b_{2n} = 0$ for all n . Similarly b_3 is properly reproduced if $a + d = -3/(4\pi^2) \approx -0.076$. This ansatz then predicts all other terms in the perturbative series:

$$\beta_{\text{pert}}^{\bar{z}} = \frac{i}{2\pi^2 y} \left(1 - \frac{3}{4\pi^2 y^2} \right)^{-1} \quad (13)$$

which gives the following coefficients b_i :

$$b_{2n} = 0, \quad b_{2n+1} = -\frac{3^n i}{2^{2n+1} \pi^{2n+2}}. \quad (14)$$

In principle, the values of a, b, c and d can be separately extracted by comparison with the leading non-perturbative terms proportional to q and \bar{q} . Writing $D = 1 - 3/(4\pi^2 y^2)$ this gives:

$$\beta_{q,\bar{q}}^z = \frac{8i}{\pi D} \left[1 - \frac{1}{2\pi y D} \left(3 + 2\pi c + \frac{3b + 2\pi a}{y} \right) \right] q - \frac{4i}{\pi^2 y D^2} \left(3 + 2\pi b + \frac{3c + 2\pi a}{y} \right) \bar{q}. \quad (15)$$

Notice, however, that the leading powers are $y^0 q$ and $y^{-1} \bar{q}$, which does not agree with ref. [8], (who finds a positive power of y premultiplying q).

The ansatz of eq. (12) has a simple zero at the fixed point, $z_* = (1 + i)/2$. The critical exponents at this point are found by diagonalizing the matrix of derivatives of the β -function at this point. Using $\mathcal{E}(z) \approx -6.10i(z - z_*) + O((z - z_*)^2)$ and $\mathcal{H}(x, y) \approx -3.69i(x - x_*) + 2.41(y - y_*) + \dots$, we find the localization length exponent $\nu \approx d/0.147$ and irrelevant exponent (see ref. [17] for definitions and a review of experimental results) $y \approx -0.096/d$ [18]. Choosing the parameter $d \approx 0.34$ puts the prediction for ν in agreement with experimental results $\nu_{\text{exp}} = 2.3 \pm 0.1$ [19] (2.4 ± 0.2 [20]).

This choice for d permits an absolute prediction from this ansatz for the irrelevant exponent: $y \approx -0.29$, which does not seem to reproduce the results of numerical simulations, which give $\nu_{\text{num}} = 2.35 \pm 0.03$ [21], and $y_{\text{num}} = -0.38 \pm 0.02$ [22] (-0.42 ± 0.04 [23]).

Ansatz 2: More complicated ansätze are also possible. For example, if $d = 0$, then R must be chosen to vanish at z_* in order to cancel the pole in $\mathcal{H}/\mathcal{D}|_{d=0}$. This is easily arranged since $f - 1/4$ has its only (double) zero at this point.

This type of ansatz contains the one proposed in ref. [5] as the special case $b = -c = A$ and $a = -A^2$, with R given by the rational function $R = (Q - 1/2)/Q$, and $Q = f + \bar{f} + 2(f - \bar{f})/(\pi A)$. The value of the parameter $A \approx 0.623$ is chosen to ensure the cancellation of the pole at z_* .

Although for holomorphic functions such a rational form for R follows on general grounds [12,13], we are not aware of any similar result for the nonholomorphic functions considered here.

This particular ansatz also does not agree, in the strong sense, with the sigma-model result at large y . This is because $R = 1 + O(q, \bar{q})$ as $z \rightarrow i\infty$, and so its prediction for the perturbative β -function is the same as the perturbative part of eq. (12), with $b + c = 0$ and $a + d = a = -A^2 \approx -0.388$. This clearly *disagrees* with the perturbative theory already at order $O(1/y^3)$. If only agreement in the weak sense is desired, then there is no reason to set $b + c = 0$, exhibiting this ansatz as one of a several-parameter family.

Putting aside agreement with sigma-model perturbation theory, the predictions for the critical exponents ob-

tained from this ansatz in ref. [5] become $\nu = 2.12$ and $y = -0.31$, which is consistent (within the roughly 10% errors) with experimental scaling data, but has difficulty with the numerical simulations of the quantum Hall system [21,22].

As was already pointed out in ref. [5], this ansatz varies as $y^0 q$ and so, like all of the previous ansätze, disagrees with the leading nonperturbative terms predicted by the sigma-model. One might wonder if the requirement of agreement with the sigma model in the very strong sense could itself be the remaining condition which uniquely determines the form of β^z . Leading instanton correction of the form $y^k q$ for $k = 1$ or $k = 2$, as required by the very strong asymptotic condition if the sigma-model instanton calculation [8] is taken seriously, can be achieved within the framework of the ansätze considered here by adding terms like $F = y^2 \mathcal{E}/f \rightarrow y^2 q + \dots$ or $G = y^2 \mathcal{H}/f \rightarrow y q + \dots$ to the product WR . Both F and G are weight $(0, -2)$ functions with simple zeros at z_* (and at $z = 0$), and so would change the values which are inferred for the critical exponents.

It would be encouraging to hope that concentration on this condition, or a solution to the consistency conditions for the metric G_{ij} discussed in ref. [1], might lead to further progress in finding sufficient conditions for the determination of the exact scaling properties of the quantum Hall system.

VI. ACKNOWLEDGEMENTS

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